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Inherited twistor-space structure of gravity amplitudes

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Abstract: At tree-level, gravity amplitudes are obtainable directly from gauge theory amplitudes via the Kawai, Lewellen and Tye closed-open string relations. We explain how the unitarity method allows us to use these relations to obtain coefficients of box integrals appearing in one-loop $\mathcal{N} = 8$ supergravity amplitudes from the recent computation of the coefficients for $\mathcal{N} = 4$ super-Yang-Mills non-maximally-helicity-violating amplitudes. We argue from factorisation that these box coefficients determine the one-loop $\mathcal{N} = 8$ supergravity amplitudes, although this remains to be proven. We also show that twistor-space properties of the $\mathcal{N} = 8$ supergravity amplitudes are inherited from the corresponding properties of $\mathcal{N} = 4$ super-Yang-Mills theory. We give a number of examples illustrating these ideas.

Keywords: Models of Quantum Gravity, Supergravity Models.

Contents

1. Introduction

Scattering amplitudes in gravity theories are closely related to those of gauge theory. At tree level there exists a set of general relations expressing gravity tree amplitudes as sums of products of gauge theory ones. These relations follow from the low energy limit of the Kawai-Lewellen-Tye (KLT) relations between open and closed string theory amplitudes [1–3]. In this limit the string relations reduces to relations for effective field theories of gravity [4, 5]. Moreover, in the low energy limit these relations do not require the existence of an underlying consistent string theory and hold in any dimensions or massless matter contents [6]. The relations also hold for large classes of higher dimension operators in the effective field theory [7].

At loop level, the standard methods for constructing amplitudes via Feynman rules provide no obvious means of exploiting the KLT relations. An alternative is provided by the unitarity method of Dixon, Kosower and two of the authors [8–11]. This method is ideal for exploiting the KLT relations at loop level, since the method obtains loop amplitudes directly from tree amplitudes, which do satisfy the KLT relations. More generally when coupled with the KLT relations, the unitarity method allows advances in gauge theory loop calculations to be carried over to gravity calculations. These ideas have been used to compute the MHV amplitudes of pure gravity [12] and $\mathcal{N}=8$ supergravity [13] and to demonstrate that $\mathcal{N}=8$ supergravity [14] is less divergent in the ultraviolet than had been expected [15, 16] previously. Recently, there have been significant advances in gauge theory computations, stimulated by Witten's proposal of a twistor-space topological string theory [17 − 22] as a candidate for a weak-weak duality to maximally supersymmetric gauge theory. This string theory generalises Nair's earlier description [23] of the simplest gauge-theory amplitudes. The twistor-space structure of the amplitudes implies that gauge theory amplitudes are simpler than had been suspected previously. In particular, for massless gauge theories, Cachazo, Svrček and Witten (CSW) [24] have presented a set of new computational rules for scattering amplitudes, in terms of "MHV vertices" inspired by the twistor space structure. Other simple versions of tree amplitudes may be found in ref. [25 − 30].

At loop level, a direct topological-string approach appears to be problematic because of the appearance of non-unitary states from conformal supergravity [31]. Nevertheless, significant progress has been accomplished at loop level by more direct means. An important step, clarifying the structure of loop amplitude, is the computation by Brandhuber, Spence and Travaglini [32] of the $\mathcal{N}=4$ MHV amplitudes using MHV vertices. Since then there has been rapid progress in obtaining amplitudes in $\mathcal{N}=4$ and $\mathcal{N}=1$ theories [33 − 37, 27, 38 − 43], including the recent computation of all $\mathcal{N}=4$ next-to-MHV (NMHV) one-loop amplitudes [28] and the next-to-next-to-MHV (N^2MHV) box coefficient [30]. These calculations rely on the four-dimensional cut constructibility of the amplitudes [8, 9] and on the knowledge of having a basis of dimensionally regularised integrals [44]. An important recent development, enhancing the power of the unitarity method, is the observation by Britto, Cachazo and Feng [30] that box integral coefficients can be obtained from generalised unitarity cuts [45, 46, 11, 27] by solving the on-shellness constraints in signature $(--++)$.

The situation with gravity is less clear. As explained in Witten's original paper [17], gravity amplitudes will have a derivative of a δ-function support rather than a simple δ-function support on degenerate curves in twistor space. This apparently complicates their structure, preventing the construction of MHV vertices for gravity amplitudes [47, 48]. Nevertheless, the KLT relations provide direct means for obtaining gravity tree amplitudes from any newly computed gauge theory tree amplitudes.

In this paper, we follow the logic of refs. [15, 12, 13], and use the unitarity method to carry over the recent gauge theory advances to quantum gravity. At one loop, supergravity is ultraviolet finite, so we do not have to concern ourselves with issues of non-renormalisability. In ref. [15] Dixon, Perelstein, Rozowsky and two of the authors observed that for four-point $\mathcal{N}=8$ supergravity, the coefficients of integral functions are proportional to products of gauge theory integral coefficients. In this paper we generalise this notion to the box coefficient of any one-loop gravity amplitude and show that in general the box integral coefficients can be expressed in terms of products of box coefficients appearing in the gauge theory. This relation between box integral coefficients holds in any theory (assuming

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we are working in a basis of only $D = 4 - 2\epsilon$ integral functions) and does not rely on the four-dimensional cut constructibility of the theory. The case we consider here explicitly is $\mathcal{N} = 8$ supergravity. The MHV amplitudes of this theory have been worked out in ref. [13].

Although the $\mathcal{N} = 8$ theory does not appear to satisfy the power counting criterion required for four-dimensional cut constructibility, we argue that the four-dimensional cuts determine the one-loop amplitudes, as suggested by their factorisation properties. Moreover, we expect that one-loop $\mathcal{N} = 8$ amplitudes are composed solely of the box integral functions. This may seem rather surprising, given the apparent violation of the cut-constructibility power-counting criterion. However, in ref. [13], explicit computations proved that up to six points in the MHV case, this miracle indeed happens. Furthermore, as argued in that reference, the factorisation properties suggests that these miracles should continue as the number of external legs increases. Here we apply the same logic to argue that even for non-MHV amplitudes we do not expect triangle, bubble or additional rational functions to appear in any $\mathcal{N} = 8$ one-loop amplitude. If true, with this ansatz we can obtain full $\mathcal{N} = 8$ one-loop amplitudes rather directly from corresponding $\mathcal{N} = 4$ super-Yang-Mills amplitudes, by relating the $\mathcal{N} = 8$ supergravity box coefficients to the $\mathcal{N} = 4$ super-Yang-Mills box coefficients.

We also use the explicitly computed results to explore the twistor-space structure of gravity amplitudes at both tree and loop level. Not too surprisingly, given the close relationship of gravity and gauge theory amplitudes, we find that the gauge theory twistor properties induce closely corresponding twistor properties on gravity amplitudes. The key difference, as already observed by Witten, is instead of δ-function support, twistor-space gravity amplitudes have a derivative of a δ-function support. At tree-level the twistor properties follow directly from analysing the amplitudes obtained via the KLT relations. At loop level we combine the unitarity method with the KLT relations in order to deduce the twistor-space properties of the coefficients of box integrals in gravity theories.

2. The Kawai-Lewellen-Tye relations

Gravity amplitudes can be constructed through the KLT-relations which connects the amplitudes for closed and open strings. The general n-point scattering amplitude for a closed string is connected to that of the open string through the following formula [1]:

$$\mathcal{M}_n^{(\text{closed string})} \sim \sum_{\Pi,\tilde{\Pi}} e^{i\pi\Phi(\Pi,\tilde{\Pi})} \mathcal{A}_n^{\text{left (open string)}}(\Pi)\mathcal{A}_n^{\text{right (open string)}}(\tilde{\Pi})\,, \tag{2.1}$$

where $\Pi(1,2,\ldots,n)$ and $\tilde{\Pi}(1,2,\ldots,n)$ are sets of the external lines of the open string modes associated with particular cyclic orderings. At infinite string tension, ($\alpha' \longrightarrow 0$) the KLT-relationship relates the field theory tree amplitudes of Yang-Mills theory and gravity. In ref. [7] this was extended also to higher derivative effective field theories of gravity.

The explicit form of the KLT-relationship up to six points at $\alpha' = 0$ is,

$$\begin{aligned} M_3^{\text{tree}}(1,2,3) &= -iA_3^{\text{tree}}(1,2,3)A_3^{\text{tree}}(1,2,3)\,,\\ M_4^{\text{tree}}(1,2,3,4) &= -is_{12}A_4^{\text{tree}}(1,2,3,4)A_4^{\text{tree}}(1,2,4,3)\,, \end{aligned}$$

$$
\begin{aligned}
M_5^{\text{tree}}(1,2,3,4,5) &= i s_{12} s_{34}\; A_5^{\text{tree}}(1,2,3,4,5) A_5^{\text{tree}}(2,1,4,3,5) + \\
&\quad + i s_{13} s_{24}\; A_5^{\text{tree}}(1,3,2,4,5) A_5^{\text{tree}}(3,1,4,2,5)\,, \\
M_6^{\text{tree}}(1,2,3,4,5,6) &= -i s_{12} s_{45}\; A_6^{\text{tree}}(1,2,3,4,5,6)(s_{35} A_6^{\text{tree}}(2,1,5,3,4,6) + \\
&\qquad\qquad + (s_{34}+s_{35})\; A_6^{\text{tree}}(2,1,5,4,3,6)) + \\
&\quad + \mathcal{P}(2,3,4)\,,
\end{aligned}
\tag{2.2}
$$

where $s_{ij} = (k_i + k_j)^2$, $\mathcal{P}(2,3,4)$ represents the sum over permutations of legs $2,3,4$ and the A_n^{tree} are tree-level colour-ordered gauge theory partial amplitudes [49 − 51]. The complete gauge theory amplitudes are obtained by multiplying these by the colour structures and by summing over permutations. In general throughout the paper, gauge theory amplitudes will be denoted by A and gravity ones by M. The KLT relations have been explicitly presented for an arbitrary number of legs [13] and these combine to give the full amplitudes via,

$$
\mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \left(\frac{\kappa}{2}\right)^{(n-2)} M_n^{\text{tree}}(1,2,\ldots,n)\,, \tag{2.3}
$$

$$
\mathcal{A}_n^{\text{tree}}(1,2,\ldots,n) = g^{(n-2)} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}\,(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}})\, A_n^{\text{tree}}(\sigma(1),\sigma(2),\ldots,\sigma(n))\,,
$$

where S_n/Z_n is the set of all permutations, but with cyclic rotations removed. The T^{a_i} are fundamental representation matrices for the Yang-Mills gauge group $\mathrm{SU}(N_c)$, normalised so that $\mathrm{Tr}(T^a T^b) = \delta^{ab}$. (For more detail on the tree and one-loop colour ordering of gauge theory amplitudes see refs. [49, 51].)

3. Unitarity method

In the unitarity-based method, loop amplitudes are constructed from tree amplitudes by considering the various cases where internal propagators go on shell. Letting two propagators go on shell is equivalent to evaluating a phase space integral over products of tree amplitudes,

$$
C_{i,\ldots,j} \equiv \frac{i}{2} \int d\mathrm{LIPS}\Big[A^{\text{tree}}(\ell_1, i, i+1, \ldots, j, \ell_2) A^{\text{tree}}(-\ell_2, j+1, j+2, \ldots, i-1, -\ell_1)\Big]\,. \tag{3.1}
$$

This phase space integral gives the discontinuity of the amplitude in the cut channel. In general, we may expand the amplitude as sum of dimensionally regularized integral functions with rational coefficients [8, 9, 44],

$$
A^{\text{1-loop}} = \sum_a \hat{c}_a I_a\,. \tag{3.2}
$$

We may then obtain the rational coefficients c_a from the cuts of the one-loop amplitude

$$
\mathrm{Im}_{K_{i,\ldots,j}>0}\, A^{\text{1-loop}} = \sum_a \hat{c}_a\, \mathrm{Im}_{K_{i,\ldots,j}>0}(I_a)\,, \tag{3.3}
$$

and their generalizations [8 − 11, 46, 11, 27, 30].

We will use two representations of the integral functions. Firstly, the scalar box integrals, I, as defined in ref. [44] and secondly the rescaled box-functions denoted F,

$$I_4 = \frac{1}{D} F\,, \tag{3.4}$$

where D is a kinematic denominator quadratic in momentum invariants. Explicitly, from ref. [8],

$$\begin{aligned} I_{4:i}^{1\mathrm{m}} &= \frac{(-2r_\Gamma)F_{n:i}^{1\mathrm{m}}}{t_{i-3}^{[2]}t_{i-2}^{[2]}}\,, \qquad I_{4:r;i}^{2\mathrm{m}e} = \frac{(-2r_\Gamma)F_{n:r;i}^{2\mathrm{m}\,e}}{t_{i-1}^{[r+1]}t_{i}^{[r+1]} - t_{i}^{[r]}t_{i+r+1}^{[n-r-2]}}\,, \qquad I_{4:r;i}^{2\mathrm{m}h} = \frac{(-2r_\Gamma)F_{n:r;i}^{2\mathrm{m}\,h}}{t_{i-2}^{[2]}t_{i-1}^{[r+1]}}\,, \\ I_{4:r,r',i}^{3\mathrm{m}} &= \frac{(-2r_\Gamma)F_{n:r,r';i}^{3\mathrm{m}}}{t_{i-1}^{[r+1]}t_{i}^{[r+r']} - t_{i}^{[r]}t_{i+r+r'}^{[n-r-r'-1]}}\,, \qquad I_{4:r,r',r'',i}^{4\mathrm{m}} = \frac{(-2r_\Gamma)F_{n:r,r',r'';i}^{4\mathrm{m}}}{t_{i}^{[r+r']}\,t_{i+r}^{[r'+r'']}\,\rho}\,, \end{aligned} \tag{3.5}$$

where we use the notation of that reference. In particular, $t_i^{[r]} = (k_i + \cdots + k_{i+r-1})^2$. (See the first appendix of ref. [8] for definitions and a more detailed description of the integral functions.) We shall use $\hat{c}_a$ for coefficients of I and c_a for coefficients of F. We can move between the two representation using

$$\hat{c}_a = c_a D\,, \tag{3.6}$$

where the kinematic denominator D for each type of integral may be read off by comparing eq. (3.4) and eq. (3.5).

One-loop massless amplitudes, which satisfy the power counting criterion that n-point Feynman integrals have no more than $n-2$ powers of loop momenta in their numerators, can be obtained directly from four-dimensional tree amplitudes [9]. Hence, when this criterion is satisfied, one may fix all rational functions appearing in the amplitudes directly from the terms which contain cuts in four dimensions. We refer to such amplitudes as "cut-constructible". Supersymmetric gauge theory amplitudes, in particular, satisfy this criterion and in the case of $\mathcal{N}=4$ super-Yang-Mills theory, the amplitude is expressible entirely as a linear combination of box integral functions [8].

A key property that allows us to relate the coefficients of box integrals in the $\mathcal{N}=4$ super-Yang-Mills theory to those of $\mathcal{N}=8$ supergravity is that the coefficients of the integrals can be determined by purely algebraic means starting from the unitarity cuts. The integral reduction method of van Neerven and Vermaseren [52] is an example of a reduction formalism with the property that in a given cut we may algebraically link the coefficients of the integrals to the original expressions for the cuts. Alternatively, it is more convenient to use generalised cuts [45, 46, 11, 27], where multiple propagators go on shell. Using the recent observation [30], that box integral coefficients can be directly obtained algebraically from generalised quadruple cuts one can straightforwardly solve for the coefficients. Since a given generalised quadruple cut selects out a unique box integral function, we may relate gravity and Yang-Mills coefficients via the KLT tree level relations, since the cuts are expressed in terms of tree amplitudes. Specifically, if weconsider an

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amplitude containing the scalar box integral function shown in figure 1, then the coefficient of this function is given by the product of the four tree amplitudes where the cut legs fully satisfy on-shell conditions [30],

$$\hat{c} = \frac{1}{2}\sum_{\mathcal{S}}\Big(A^{\rm tree}(\ell_1, i_1, \ldots, i_2, \ell_2) \times A^{\rm tree}(\ell_2, i_3, \ldots, i_4, \ell_3) \times \qquad (3.7)$$
$$\times A^{\rm tree}(\ell_3, i_5, \ldots, i_6, \ell_4) \times A^{\rm tree}(\ell_4, i_7, \ldots, i_8, \ell_1)\Big)\,,$$

where $\mathcal{S}$ indicates the set of helicity configurations of the legs ℓ_i which give a non-vanishing product of tree amplitudes. Employing signature $(--++)$ or complex momenta is useful in this context because it allows one to use this formula even when one of the tree amplitudes is a three-point amplitude: In Minkowski signature with real momenta such on-shell tree amplitudes vanish.

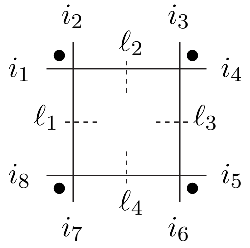

Figure 1: A quadruple cut of a n-point amplitude. The dashed lines represent the cuts. The dot represents an arbitrary number of external line insertions.

4. Gauge theory results

In this section we will collect various results in $\mathcal{N}=4$ gauge theory that we use in section 6 to obtain gravity one-loop amplitudes.

4.1 MHV amplitudes

The $\mathcal{N}=4$ MHV amplitudes are remarkably simple and were first calculated in ref. [8]. They have also been re-computed using twistor-inspired methods [32, 36] and are given by simple linear combinations of box integral functions,

$$A_{n;1}^{\mathcal{N}=4\ \rm MHV} = \hat{r}_\Gamma\, A_n^{\rm tree} \times V_n^g\,. \qquad (4.1)$$

The factor V_n^g $(n \geq 5)$ depends on whether n is odd $(n = 2m+1)$ or even $(n = 2m)$,

$$\begin{aligned}
(\mu^2)^{-\epsilon} V^g_{2m+1} &= \sum_{r=2}^{m-1}\sum_{i=1}^{n} F^{2{\rm m}\,e}\big(s_{i\ldots(i+r)}, s_{(i-1)\ldots(i+r-1)}, s_{i\ldots(i+r-1)}, s_{(i+r+1)\ldots(i-2)}\big) + \\
&\quad + \sum_{i=1}^{n} F^{1{\rm m}}\big(s_{i-3,i-2}, s_{i-2,i-1}, s_{i\ldots(i-4)}\big)\,, \\
(\mu^2)^{-\epsilon} V^g_{2m} &= \sum_{r=2}^{m-2}\sum_{i=1}^{n} F^{2{\rm m}\,e}\big(s_{i\ldots(i+r)}, s_{(i-1)\ldots(i+r-1)}, s_{i\ldots(i+r-1)}, s_{(i+r+1)\ldots(i-2)}\big) + \\
&\quad + \sum_{i=1}^{n} F^{1{\rm m}}\big(s_{i-3,i-2}, s_{i-2,i-1}, s_{i\ldots(i-4)}\big) + \qquad (4.2) \\
&\quad + \sum_{i=1}^{n/2} F^{2{\rm m}\,e}\big(s_{i\ldots(i+m-1)}, s_{(i-1)\ldots(i+m-2)}, s_{i\ldots(i+m-2)}, s_{(i+m)\ldots(i-2)}\big)\,.
\end{aligned}$$

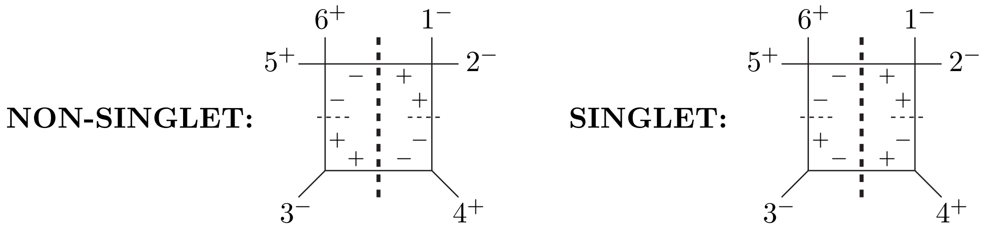

Figure 2: The non-singlet and singlet contributions to a two-mass hard box integral. The dashed line indicates the cut to which the singlet and non-singlet description refer.

using the box integral functions F as defined in eq. (3.5) and given in the first appendix of ref. [8]. In the above

$$\hat{r}_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} r_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,, \tag{4.3}$$

is a prefactor. Because of the dimensional regulator, an overall factor of $(\mu^2)^\epsilon$ enters where μ is an arbitrary scale.

4.2 NMHV amplitudes

First we consider the six-gluon amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory. These amplitudes were calculated in ref. [9]. We collect these results here, since we will use them in section 6 to obtain the corresponding $\mathcal{N} = 8$ supergravity amplitudes. There are three independent NMHV super-Yang-Mills partial amplitudes. Since the leading colour gauge theory amplitudes are colour-ordered we need only consider one cyclic ordering. (Subleading colour partial amplitudes may be obtained from the leading ones, by summing over appropriate permutations [8]).

In general, the six-point box coefficients are of the form [9],

$$c = c_{NS} + c_S\,, \tag{4.4}$$

where the two terms arise from different helicity structures in the cuts three-particle channels, as illustrated in figure 2. (In one case below one of the two terms vanishes.) A six-point box has only one of its cuts in a three-particle channel. The cut in the three particle channel may be divided into "singlet" and "non-singlet" contributions as shown in figure 2. The coefficient c_{NS} represents the non-singlet contribution and c_S the singlet contribution. The singlet term corresponds to the two cut legs having the same helicity on one side of the cut. The singlet terms thus has contributions only from gluons crossing the cut. The non-singlet term has its cut legs having opposite helicity on one side of the cut. For this configuration all terms in the $\mathcal{N} = 4$ multiplet contribute.

First consider the amplitude, $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$. In this case, there will be a total of twelve non-vanishing integral coefficients. In this case many of the coefficients are equal to each other,

$$
\begin{aligned}
c^{(1^-2^-3^-)4^+5^+6^+} &= c^{(4^+5^+6^+)1^-2^-3^-} = c^{(2^-3^-)(4^+5^+)6^+1^-} = c^{(5^+6^+)(1^-2^-)3^-4^+} = B_1\,,\\
c^{(2^-3^-4^+)5^+6^+1^-} &= c^{(5^+6^+1^-)2^-3^-4^+} = c^{(3^-4^+)(5^+6^+)1^-2^-} = c^{(6^+1^-)(2^-3^-)4^+5^+} = B_2\,,\\
c^{(3^-4^+5^+)6^+1^-2^-} &= c^{(6^+1^-2^-)3^-4^+5^+} = c^{(4^+5^+)(6^+1^-)2^-3^-} = c^{(1^-2^-)(3^-4^+)5^+6^+} = B_3\,,
\end{aligned}
\tag{4.5}
$$

where the plus and minus labels on the legs refer to the helicity labels in an all outgoing convention. For B_1 the non-singlet contribution vanishes whilst for the other two they are a sum on non-singlet and singlet contributions. From ref. [9], the explicit values are,

$$
\begin{aligned}
B_1 &= i\,\frac{(K^2)^3}{[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\;\;\langle 1^+|K|4^+\rangle\langle 3^+|K|6^+\rangle}\,, && K = K_{123}\,,\\
B_2 &= \left(\frac{\langle 4^+|K|1^+\rangle}{K^2}\right)^4 B_1|_{j\to j+1} + \left(\frac{\langle 2\,3\rangle\,[5\,6]}{K^2}\right)^4 B_1^\dagger|_{j\to j+1}\,, && K = K_{234}\,,\\
B_3 &= \left(\frac{\langle 6^+|K|3^+\rangle}{K^2}\right)^4 B_1|_{j\to j-1} + \left(\frac{\langle 1\,2\rangle\,[4\,5]}{K^2}\right)^4 B_1^\dagger|_{j\to j-1}\,, && K = K_{345}\,.
\end{aligned}
\tag{4.6}
$$

where the gluon polarisation tensors have been expressed in a spinor helicity [53, 49] basis. In this formalism amplitudes are expressed in terms of spinor inner-products,

$$
\langle j\,l\rangle = \langle j^-|l^+\rangle = \bar{u}_-(k_j)u_+(k_l)\,, \qquad [j\,l] = \langle j^+|l^-\rangle = \bar{u}_+(k_j)u_-(k_l)\,,
\tag{4.7}
$$

where $u_\pm(k)$ is a massless Weyl spinor with momentum k and plus or minus chirality. With the normalisation used here, $[i\,j] = \mathrm{sign}(k_i^0 k_j^0)\,\langle j\,i\rangle^*$ so that,

$$
\langle i\,j\rangle\,[j\,i] = 2k_i\cdot k_j = s_{ij}\,.
\tag{4.8}
$$

(Note that $[i\,j]$ defined in this way differs by an overall sign from the notation commonly used in twistor-space studies [17].)

For $A_{6;1}^{\mathcal{N}=4}(1^-,2^-,3^+,4^-,5^+,6^+)$, the box coefficients are,

$$
\begin{aligned}
c^{(1^-2^-3^+)4^-5^+6^+} &= c^{(4^-5^+6^+)1^-2^-3^+} = c^{(2^-3^+)(4^-5^+)6^+1^-} = c^{(5^+6^+)(1^-2^-)3^+4^-} = D_1\,,\\
c^{(2^-3^+4^-)5^+6^+1^-} &= c^{(5^+6^+1^-)2^-3^+4^-} = c^{(3^+4^-)(5^+6^+)1^-2^-} = c^{(6^+1^-)(2^-3^+)4^-5^+} = D_2\,,\\
c^{(3^+4^-5^+)6^+1^-2^-} &= c^{(6^+1^-2^-)3^+4^-5^+} = c^{(4^-5^+)(6^+1^-)2^-3^+} = c^{(1^-2^-)(3^+4^-)5^+6^+} = D_3\,,
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{aligned}
D_1 &= \left(\frac{\langle 3^+|K|4^+\rangle}{K^2}\right)^4 B_1 + \left(\frac{\langle 1\,2\rangle\,[5\,6]}{K^2}\right)^4 B_1^\dagger\,, && K = K_{123}\,,\\
D_2 &= \left(\tfrac{\langle 3^+|K|1^+\rangle}{K^2}\right)^4 B_1|_{j\to j+1} + \left(\tfrac{\langle 2\,4\rangle[5\,6]}{K^2}\right)^4 B_1^\dagger|_{j\to j+1}\,, && K = K_{234}\,,\\
D_3 &= \left(\tfrac{\langle 6^+|K|4^+\rangle}{K^2}\right)^4 B_1|_{j\to j-1} + \left(\tfrac{\langle 1\,2\rangle[3\,5]}{K^2_{345}}\right)^4 B_1^\dagger|_{j\to j-1}\,, && K = K_{345}\,.
\end{aligned}
\tag{4.10}
$$

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Finally, for $A_{6;1}^{\mathcal{N}=4}(1^-,2^+,3^-,4^+,5^-,6^+)$,

$$\begin{aligned}
c^{(1^-2^+3^-)4^+5^-6^+} &= c^{(4^+5^-6^+)1^-2^+3^-} = c^{(2^+3^-)(4^+5^-)6^+1^-} = c^{(5^-6^+)(1^-2^+)3^-4^+} = G_1\,,\\
c^{(2^+3^-4^+)5^-6^+1^-} &= c^{(5^-6^+1^-)2^+3^-4^+} = c^{(3^-4^+)(5^-6^+)1^-2^+} = c^{(6^+1^-)(2^+3^-)4^+5^-} = G_2\,,\\
c^{(3^-4^+5^-)6^+1^-2^+} &= c^{(6^+1^-2^+)3^-4^+5^-} = c^{(4^+5^-)(6^+1^-)2^+3^-} = c^{(1^-2^+)(3^-4^+)5^-6^+} = G_3\,,
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
G_1 &= \left(\frac{\langle 2^+|K|5^+\rangle}{K^2}\right)^4 B_1 + \left(\frac{\langle 1\,3\rangle\,[4\,6]}{K^2}\right)^4 B_1^\dagger\,, & K &= K_{123}\,,\\
G_2 &= \left(\tfrac{\langle 6^+|K|3^+\rangle}{K^2}\right)^4 B_1^\dagger|_{j\to j+1} + \left(\tfrac{\langle 5\,1\rangle[2\,4]}{K^2}\right)^4 B_1|_{j\to j+1}\,, & K &= K_{234}\,,\\
G_3 &= \left(\tfrac{\langle 4^+|K|1^+\rangle}{K^2}\right)^4 B_1^\dagger|_{j\to j-1} + \left(\tfrac{\langle 3\,5\rangle[6\,2]}{K^2}\right)^4 B_1|_{j\to j-1}\,, & K &= K_{345}\,.
\end{aligned} \tag{4.12}$$

Recently, the complete expression for all NMHV amplitudes in $\mathcal{N}=4$ super-Yang-Mills theory have been obtained using the unitarity method [28]. We will use the results of that paper to construct some examples of n-point box coefficients in the $\mathcal{N}=8$ theory with $n>6$.

5. Structure of one-loop gravity amplitudes

For simplicity, in the forthcoming equations we will define one-loop amplitudes in gravity for which all field couplings have been removed, i.e.,

$$\mathcal{M}_n^{\text{1-loop}} = \left(\frac{\kappa}{2}\right)^n M_n^{\text{1-loop}}(1,2,\ldots,n)\,. \tag{5.1}$$

In gravity theories, the three graviton vertex contains two powers of momenta. A generic n-point diagram will involve a loop momenta integral where the polynomial on the numerator is, in general, of degree $2n$ in loop momenta. In $\mathcal{N}=8$ supergravity, there are cancellations between the contributions of different particle types, and in a suitable formulation the polynomial is only of degree $2n-8$ in the loop momenta. (See, for example, the "String Based Method" [54, 55] where the cancellation is explicit in terms of Feynman parameters.) Recall, amplitudes are cut-constructible if the loop momenta polynomial is of degree $\leq n-4$ and thus we would not expect $\mathcal{N}=8$ supergravity amplitudes to be cut constructible for $n>6$. The Passarino-Veltman [56] reduction, decreases a polynomial of degree r in a n-point integral to a box integral with polynomial degree $r-(n-4)$. Hence for $n>4$ we would *a priori* expect tensor box integrals reduced to scalar boxes plus triangle and bubble integrals.

However, as demonstrated in ref. [13], the one-loop MHV amplitudes of $\mathcal{N}=8$ supergravity do have a much better power behaviour than expected and appear to satisfy the cut constructibility criterion. This was demonstrated through six points by direct calculation and argued to hold for all n based on the factorisation properties. Even more surprisingly the $\mathcal{N}=8$ supergravity amplitudes appear to obey precisely the same power counting as those of $\mathcal{N}=4$ super-Yang-Mills theory, *i.e.,* for an n-point integral there are $n-4$ powers of loop momentum. This power counting behaviour is reflected in the lack of triangle, bubble or additional rational function contributions to MHV $\mathcal{N}=8$ supergravity amplitudes.

We do not have a proof that this feature holds more generally for non-MHV amplitudes. However, we conjecture that it does hold generally for $\mathcal{N}=8$ one-loop amplitudes, using arguments involving the factorisation properties of the amplitudes. Examining their various factorisations, we find no evidence requiring integral functions other than boxes to be present in the one-loop amplitudes. In particular since the NMHV amplitudes can be reduced to MHV amplitudes in various factorisation limits either the non-box functions must all vanish in all such limits or, more likely, be absent in NMHV amplitudes.

More explicitly, consider the multi-particle factorisations. From general field theory considerations, amplitudes must factorise (up to subtleties having to do with infrared singularities) on multi-particle poles. For $K^\mu \equiv k_i^\mu + \ldots + k_{i+r+1}^\mu$ the amplitude factorises when K becomes on shell. Specifically, as $K^2 \to 0$ the factorisation properties for one-loop infrared singular amplitudes are described by [57],

$$
\begin{aligned}
M_{n;1}^{\text{1-loop}} \stackrel{K^2\to 0}{\longrightarrow} \sum_{\lambda=\pm}\Bigg[& M_{r+1;1}^{\text{1-loop}}(k_i,\ldots,k_{i+r-1},K^\lambda)\,\frac{i}{K^2}\,M_{n-r+1}^{\text{tree}}((-K)^{-\lambda},k_{i+r},\ldots,k_{i-1}) + \\
& + M_{r+1}^{\text{tree}}(k_i,\ldots,k_{i+r-1},K^\lambda)\,\frac{i}{K^2}\,M_{n-r+1;1}^{\text{1-loop}}((-K)^{-\lambda},k_{i+r},\ldots,k_{i-1}) + \\
& + M_{r+1}^{\text{tree}}(k_i,\ldots,k_{i+r-1},K^\lambda)\times \\
& \times \frac{i}{K^2}\,M_{n-r+1}^{\text{tree}}((-K)^{-\lambda},k_{i+r},\ldots,k_{i-1})\,\hat{r}_\Gamma\,\mathcal{F}_n(K^2;k_1,\ldots,k_n)\Bigg], \quad (5.2)
\end{aligned}
$$

where the one-loop "factorisation function" $\mathcal{F}_n$ is independent of helicities. Consider the multi-particle factorisation of a NMHV amplitude. In general, the two factorised amplitudes appearing on either side of the pole will be MHV amplitudes, since supersymmetry requires all amplitudes with less than two negative helicities to vanish. From ref. [13] we know that the one-loop MHV amplitudes should not contain triangle of bubble integrals. Therefore, in *all* multi-particle factorisation limits we cannot encounter these integral functions. Moreover, in any soft or collinear limit which reduces negative helicities by one, there can be no triangle or bubble functions. One can continue this bootstrap adding increasing numbers of negative helicities. Although this does not constitute a proof that triangle and bubble integrals cannot appear we know of no counterexample, with six or higher points, where a factorisation bootstrap has failed to produce the correct result. In this case, the result is that bubble and triangle integrals should not appear in one-loop $\mathcal{N}=8$ supergravity amplitudes. In the remaining part of the paper we will assume that:

> *The only non-vanishing integral functions in $\mathcal{N}=8$ one-loop amplitudes are scalar box integral functions.*

Thus, we write the $\mathcal{N}=8$ one-loop amplitudes as

$$M_n^{\mathcal{N}=8} = i\hat{r}_\Gamma(\mu^2)^\epsilon \sum c_{\mathcal{N}=8}^{(i_1\ldots i_2)(i_3\ldots i_4)(i_5\ldots i_6)(i_7\ldots i_8)} \times \\ \times F_{(i_1\ldots i_2)(i_3\ldots i_4)(i_5\ldots i_6)(i_7\ldots i_8)}\,, \tag{5.3}$$

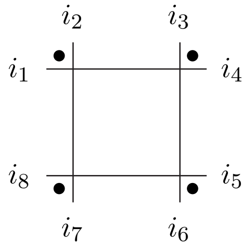


Figure 3: The labels corresponding to a box coefficient. In the gravity case there is no ordering imposed on the legs.

where the sum runs over all inequivalent box functions. The $F_{(i_1\ldots i_2)(i_3\ldots i_4)(i_5\ldots i_6)(i_7\ldots i_8)}$ are dimensionally regulated box functions: although supergravity theories are one-loop ultraviolet finite in four dimensions the amplitudes contain infrared infinities requiring regularisation. The $c_{\mathcal{N}=8}^{(i_1\ldots i_2)(i_3\ldots i_4)(i_5\ldots i_6)(i_7\ldots i_8)}$ are the kinematic coefficients, where the parenthesis indicate which legs belong in a cluster as illustrated in figure 3. In the gravity case, in contrast to the gauge theory case, the legs may appear in any ordering, since there is no colour ordering.

6. Constructing supergravity amplitudes from super-Yang-Mills amplitudes

In this section, we obtain the box-coefficients of gravity amplitudes using the KLT relations and the solutions for the Yang-Mills amplitudes. As examples, we will obtain all the box coefficients of the six-graviton amplitude in $\mathcal{N}=8$ supergravity, as well as a few selected coefficients at seven, eight and n points. The MHV n-point amplitude have been previously been obtained in ref. [13], so here we focus on NMHV amplitudes. We also discuss one N^{2}MHV example at eight points.

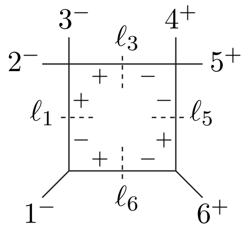
6.1 Six-graviton NMHV amplitudes

In this subsection, we obtain the box-coefficients of the NMHV $\mathcal{N}=8$ six-graviton amplitude in terms of the, known, box coefficients of $\mathcal{N}=4$ super-Yang-Mills, collected in section 4.2. There are three independent NMHV amplitudes in Yang-Mills depending on the positions of the three negative legs: $A(1^-2^-3^-4^+5^+6^+)$, $A(1^-2^-3^+4^-5^+6^+)$ and $A(1^-2^+3^-4^+5^-6^+)$. In gravity, on the other hand, there is no colour ordering and hence a single distinct amplitude: $M(1^-2^-3^-4^+5^+6^+)$. The box-coefficients of the gravity amplitude will be expressible in terms of the box coefficients of the three Yang-Mills amplitudes. For six-point amplitudes, there are three possible scalar box structures in the amplitude: the "single-mass", "two-mass-hard" and "two-mass-easy" boxes. (See, *e.g.,* the first appendix of ref. [9], for definitions of these functions in the euclidean region.) Of these only the first two contribute to the super-Yang-Mills six-gluon NMHV amplitude. This property is immediately "inherited" by the gravity case. If we evaluate the appropriate generalised cut and express the gravity amplitudes in terms of Yang-Mills we immediately find that the gravity coefficient vanishes whenever the corresponding super-Yang-Mills coefficient vanishes. For the "two-mass-hard" coefficients there are total of 45 independent boxes all of which appear in the amplitude. These can be split into four distinct cases depending on the position of the three negative legs,

$$I^{(--),(++),+-}\,, \qquad I^{(--),(++),-+}\,, \qquad I^{(--),(-+),++}\,, \qquad I^{(-+),(-+),-+}\,. \tag{6.1}$$

All other coefficients of two-mass-hard boxes are obtained by symmetry and parity conjugation.

Taking the first case it is straightforward to show that only a single helicity configuration of the internal lines contributes to the quadruple cut



In this case, gravitons are the only possible states when all cut conditions are imposed. The coefficient of the box function is thus

$$\begin{aligned}
\hat{c}^{(2^-3^-)(4^+5^+)6^+1^-}_{\mathcal{N}=8} &= \frac{1}{2}M^{\rm tree}(2^-,3^-,-\ell_1^+,\ell_3^+)\times M^{\rm tree}(4^+,5^+,-\ell_3^-,\ell_5^-)\times\\
&\quad \times M^{\rm tree}(6^+,-\ell_5^+,\ell_6^-)\times M^{\rm tree}(1^-,-\ell_6^+,\ell_1^-)\times\\
&= \frac{1}{2}s_{23}A^{\rm tree}(2^-,3^-,-\ell_1^+,\ell_3^+)A^{\rm tree}(3^-,2^-,-\ell_1^+,\ell_3^+)\times\\
&\quad \times s_{45}A^{\rm tree}(4^+,5^+,-\ell_3^-,\ell_5^-)A^{\rm tree}(5^+,4^+,-\ell_3^-,\ell_5^-)\times\\
&\quad \times A^{\rm tree}(6^+,-\ell_5^+,\ell_6^-)A^{\rm tree}(6^+,-\ell_5^+,\ell_6^-)\times\\
&\quad \times A^{\rm tree}(1^-,-\ell_6^+,\ell_1^-)A^{\rm tree}(1^-,-\ell_6^+,\ell_1^-)\\
&= \frac{1}{2}s_{23}s_{45}\times\Big(A^{\rm tree}(2^-,3^-,-\ell_1^+,\ell_3^+)\times A^{\rm tree}(4^+,5^+,-\ell_3^-,\ell_5^-)\times\\
&\qquad\qquad \times A^{\rm tree}(6^+,-\ell_5^+,\ell_6^-)\times A^{\rm tree}(1^-,-\ell_6^+,\ell_1^-)\Big)\times\\
&\quad \times\Big(A^{\rm tree}(3^-,2^-,-\ell_1^+,\ell_3^+)\times A^{\rm tree}(5^+,4^+,-\ell_3^-,\ell_5^-)\times\\
&\qquad \times A^{\rm tree}(6^+,-\ell_5^+,\ell_6^-)\times A^{\rm tree}(1^-,-\ell_6^+,\ell_1^-)\Big)\\
&= 2s_{23}s_{45}\times\hat{c}_S^{(2^-3^-)(4^+5^+)6^+1^-}\times\hat{c}_S^{(3^-2^-)(5^+4^+)6^+1^-}\,,
\end{aligned}\tag{6.2}$$

where the $\mathcal{N}=4$ box integral coefficients $\hat{c}$ are defined in subsection 4.2. In going to the last line, we used the fact that the quadruple cut freezes the loop integral [30], determining the $\mathcal{N}=4$ coefficients directly. After substituting for the explicit form of the $\hat{c}$, using eqs. (3.6) and (4.5) and relabeling, we obtain,

$$\begin{aligned}
&\hat{c}^{(2^-3^-)(4^+5^+)6^+1^-}_{\mathcal{N}=8} =\\
&\quad = \frac{i}{2}\frac{s_{23}s_{45}s_{61}^2(K_{123}^2)^8}{[1\,2]\,[2\,3]\,[1\,3]\,[3\,2]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 5\,4\rangle\,\langle 4\,6\rangle\,\langle 1^+|K|4^+\rangle\langle 1^+|K|5^+\rangle\langle 2^+|K|6^+\rangle\langle 3^+|K|6^+\rangle}\,.
\end{aligned}$$

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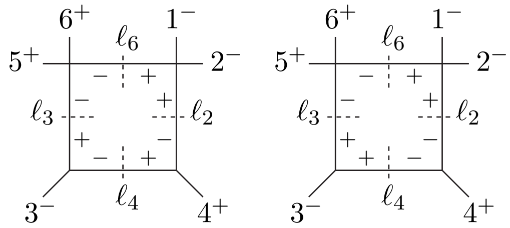
Consider now the coefficient $\hat{c}_{\mathcal{N}=8}^{(5^+6^+)(1^-2^-)4^+3^-}$. Here there are two possible helicity configurations in the generalized cuts,



where, again, only gravitons may contribute. For the Yang-Mills case the two configurations give rise to the non-singlet and singlet terms in the box coefficients. For the gravity case we take each configuration separately and decompose the gravity amplitudes into Yang-Mills amplitudes. The gravity coefficient will then be

$$\hat{c}_{\mathcal{N}=8}^{(5^+6^+)(1^-2^-)4^+3^-} = 2s_{56}s_{12}\left(\sum_{i=NS,S} \hat{c}_i^{(5^+6^+)(1^-2^-)4^+3^-} \times \hat{c}_i^{(6^+5^+)(2^-1^-)4^+3^-}\right). \tag{6.3}$$

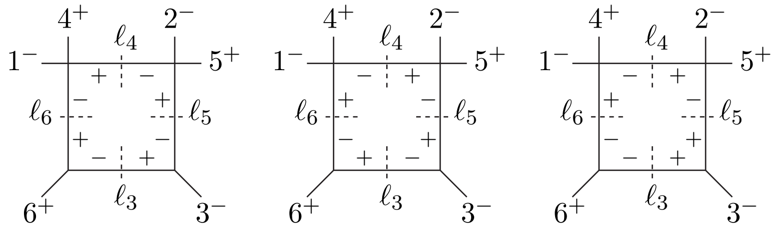
Explicitly, this evaluates to

$$\begin{aligned}&\frac{i}{2}\frac{\langle 4^+|K|3^+\rangle^8 s_{12}s_{56}s_{34}^2}{\langle 3\,5\rangle\langle 3\,6\rangle\langle 5\,6\rangle\langle 6\,5\rangle[1\,2][1\,4][2\,1][2\,4]\langle 1^+|K|3^+\rangle\langle 2^+|K|3^+\rangle\langle 4^+|K|5^+\rangle\langle 4^+|K|6^+\rangle}+\\ &+\frac{i}{2}\frac{\langle 1\,2\rangle^6[5\,6]^6 s_{12}s_{56}s_{34}^2}{\langle 1\,4\rangle\langle 2\,4\rangle[3\,5][3\,6]\langle 3^+|K|1^+\rangle\langle 3^+|K|2^+\rangle\langle 5^+|K|4^+\rangle\langle 6^+|K|4^+\rangle}; \qquad K=K_{124}\,,\end{aligned} \tag{6.4}$$

Similarly we have

$$\begin{aligned}\hat{c}_{\mathcal{N}=8}^{(2^-3^-)(4^+1^-)5^+6^+} &= 2s_{23}s_{41}\times\left(\sum_{i=NS,S}\hat{c}_i^{(2^-3^-)(4^+1^-)5^+6^+}\times\hat{c}_i^{(3^-2^-)(1^-4^+)5^+6^+}\right),\\ \hat{c}_{\mathcal{N}=8}^{(3^-4^+)(5^+6^+)1^-2^-} &= 2s_{34}s_{56}\times\left(\sum_{i=NS,S}\hat{c}_i^{(3^-4^+)(5^+6^+)1^-2^-}\times\hat{c}_i^{(4^+3^-)(6^+5^+)1^-2^-}\right).\end{aligned} \tag{6.5}$$

The final two-mass box case is $\hat{c}_{\mathcal{N}=8}^{(1^-4^+)(2^-5^+)3^-6^+}$. In this case there are three possible solutions,



In this example, we must also consider the contributions from all the states of the $\mathcal{N}=8$ multiplet to the first and second diagrams. The combination of these different particle contributions follows closely from the Yang-Mills case. For the $\mathcal{N}=4$ super-Yang-Mills case the MHV tree amplitudes for states of different helicity can be related to the tree amplitude with two gluons and two scalars by, for example,

$$A^{\rm tree}(1^+,2^-,\ell_1^h,\ell_2^{-h}) = x^{2h} A^{\rm tree}(1^+,2^-,\ell_1^s,\ell_2^{-s})\,, \tag{6.6}$$

by application of supersymmetric Ward identities [58]. By applying the supersymmetric Ward identities at each corner the contributions to the quadruple cut from each particle type will be related to that of the scalar by an overall factor of X^{2h} where

$$X = x_1 x_2 x_3 x_4\,. \tag{6.7}$$

Summing over all the possible helicities over the first and second configurations gives a total coefficient, which is that of the scalar multiplied by

$$\rho_{\mathcal{N}=4} = \sum_{h\in H} X^{2h} = \frac{(X-1)^4}{X^2}\,. \tag{6.8}$$

When we carry out the same procedure for $\mathcal{N}=8$ supergravity the argument follows in a very similar way (see ref. [15] for the details of summing over the $\mathcal{N}=8$ multiplet) with the contributions from the entire multiplet, summed over the two configurations, being that of the scalar times

$$\rho_{\mathcal{N}=8} = (\rho_{\mathcal{N}=4})^2\,. \tag{6.9}$$

This gives this contribution to the box coefficient as the product of the non-singlet Yang-Mills terms. The third solution gives a contribution which is the product of the Yang-Mills singlet terms so that, adding the contributions together gives

$$\hat c_{\mathcal{N}=8}^{(1^-4^+)(2^-5^+)3^-6^+} = 2 s_{14} s_{25} \times \left(\sum_{i=NS,S} \hat c_i^{(1^-4^+)(2^-5^+)3^-6^+} \times \hat c_i^{(4^+1^-)(5^+2^-)3^-6^+} \right). \tag{6.10}$$

In general, if we consider boxes with massive legs containing more than two external legs, then the KLT relationships will express the gravity tree as a sum of products of Yang-Mills trees. Inserting this into the generalised cuts the $\mathcal{N}=8$ box coefficient will be a sum over products of $\mathcal{N}=4$ coefficients

$$\hat c_{\mathcal{N}=8} = \sum P(s_{ij}) \hat c_{\mathcal{N}=4}\, \hat c'_{\mathcal{N}=4}\,, \tag{6.11}$$

where $P(s_{ij})$ is a polynomial in the s_{ij}.

Continuing with the six-point amplitude we now consider the coefficients of the "one-mass" box. There are a total of 60 such box functions. These split into four classes depending on whether the massive leg contains three, two, one or no negative helicities. Considering the case of the massive leg containing three negative helicities, we have

$$\begin{aligned} \hat c_{\mathcal{N}=8}^{(1^-2^-3^-)4^+5^+6^+} &= \frac{1}{2} M^{\rm tree}(1^-,2^-,3^-,-\ell_6^+,\ell_3^+) \times M^{\rm tree}(4^+,-\ell_3^-,\ell_4^+) \times \\ &\times \times M^{\rm tree}(5^+,-\ell_4^-,\ell_5^-) \times M^{\rm tree}(6^+,-\ell_5^+,\ell_6^-)\,. \end{aligned} \tag{6.12}$$

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Now the KLT expansion for $M^{\rm tree}(1^-,2^-,3^-,-\ell_6^+,\ell_3^+)$ gives

$$\begin{aligned} M^{\rm tree}(1^-,2^-,3^-,\ell_3^+,-\ell_6^+) &= i s_{12} s_{3\ell_3} A^{\rm tree}(1^-,2^-,3^-,\ell_3^+,-\ell_6^+) A^{\rm tree}(2^-,1^-,\ell_3^+,3^-,-\ell_6^+) + \\ &\quad + i s_{13} s_{2\ell_3} A^{\rm tree}(1^-,3^-,2^-,\ell_3^+,-\ell_6^+) A^{\rm tree}(3^-,1^-,\ell_3^+,2^-,-\ell_6^+)\,. \end{aligned} \tag{6.13}$$

This expression contains Yang-Mills amplitudes where the cut legs ℓ_6 and ℓ_3 are not adjacent. To recombine these into Yang-Mills box coefficients we must remedy this by using the "decoupling identity" among colour-ordered tree amplitudes [51]

$$A^{\rm tree}(a,\{\alpha\},b,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in OP\{\alpha\}\{\beta^T\}} A^{\rm tree}(a,\sigma(\{\alpha\}\{\beta^T\}),b)\,, \tag{6.14}$$

where n_β is the number of elements in set $\{\beta\}$, β^T is the set β with ordering reversed, and $OP\{\alpha\}\{\beta^T\}$ is the set of permutation of $\{\alpha\}\cup\{\beta^T\}$ preserving the ordering of elements within each of the two sets. The decoupling identity, with $a=\ell_3, b=-\ell_6, \{\alpha\}=\{1\}$ and $\{\beta\}=\{2,1\}$, implies

$$\begin{aligned} A^{\rm tree}(2^-,1^-,\ell_3^+,3^-,-\ell_6^+) &= A^{\rm tree}(\ell_3^+,3^-,-\ell_6^+,2^-,1^-) \\ &= \Big(A^{\rm tree}(\ell_3^+,3^-,1^-,2^-,-\ell_6^+) + A^{\rm tree}(\ell_3^+,1^-,3^-,2^-,-\ell_6^+) + \\ &\qquad + A^{\rm tree}(\ell_3^+,1^-,2^-,3^-,-\ell_6^+)\Big)\,, \\ A^{\rm tree}(3^-,1^-,\ell_3^+,2^-,-\ell_6^+) &= A^{\rm tree}(\ell_3^+,2^-,-\ell_6^+,3^-,1^-) \\ &= \Big(A^{\rm tree}(\ell_3^+,2^-,1^-,3^-,-\ell_6^+) + A^{\rm tree}(\ell_3^+,1^-,2^-,3^-,-\ell_6^+) + \\ &\qquad + A^{\rm tree}(\ell_3^+,1^-,3^-,2^-,-\ell_6^+)\Big)\,. \end{aligned} \tag{6.15}$$

Recombining the products of Yang-Mills tree amplitudes into Yang-Mills box coefficients, we obtain for the gravity box coefficient

$$\begin{aligned} \hat c_{\mathcal{N}=8}^{(1^-2^-3^-)4^+5^+6^+} &= 2 s_{12} s_{3\ell_3}\Big(\hat c_S^{(1^-2^-3^-)4^+5^+6^+}\,\hat c_S^{(2^-1^-3^-)4^+5^+6^+} + \\ &\qquad + \hat c_S^{(1^-2^-3^-)4^+5^+6^+}\,\hat c_S^{(2^-3^-1^-)4^+5^+6^+} + \\ &\qquad + \hat c_S^{(1^-2^-3^-)4^+5^+6^+}\,\hat c_S^{(3^-2^-1^-)4^+5^+6^+}\Big) \\ &\quad + 2 s_{13} s_{2\ell_3}\Big(\hat c_S^{(1^-3^-2^-)4^+5^+6^+}\,\hat c_S^{(3^-1^-2^-)4^+5^+6^+} + \\ &\qquad + \hat c_S^{(1^-3^-2^-)4^+5^+6^+}\,\hat c_S^{(3^-2^-1^-)4^+5^+6^+} + \\ &\qquad + \hat c_S^{(1^-3^-2^-)4^+5^+6^+}\,\hat c_S^{(2^-3^-1^-)4^+5^+6^+}\Big)\,, \end{aligned} \tag{6.16}$$

where we replace

$$s_{a\ell_3} = \frac{s_{56}\langle 5^+|a|4^+\rangle}{[5\,6]\,\langle 6\,4\rangle} + s_{a4}\,, \tag{6.17}$$

which is obtained by applying the on-shell conditions to the generalised cuts [30]. (If the massless corner attached to leg four (4) was a "mostly-minus" three point amplitude rather than a "mostly-plus" amplitudes then solving the on-shell conditions gives a formula for $s_{a\ell_3}$ which is the complex conjugate of the above.) The other one-mass box coefficients obey an analogous formulæ involving a summation over the singlet and non-singlet solutions.

Summarising, we have shown that the $\mathcal{N}=8$ coefficients are given, in terms of the $\mathcal{N}=4$ super-Yang-Mills coefficients, by

$$\begin{aligned}
\hat{c}^{(ab)c(de)f}_{N=8} &= 0\,, \\
\hat{c}^{(ab)(cd)ef}_{N=8} &= 2 s_{ab} s_{cd} \times \left(\sum_{i=NS,S} \hat{c}^{(ab)(cd)ef}_i \times \hat{c}^{(ba)(dc)ef}_i \right), \\
\hat{c}^{(abc)def}_{N=8} &= 2 s_{ab} s_{c\ell_c} \sum_{i=NS,S} \left(\hat{c}^{(abc)def}_i \hat{c}^{(bac)def}_i + \hat{c}^{(abc)def}_i \hat{c}^{(bca)def}_i + \hat{c}^{(abc)def}_i \hat{c}^{(cba)def}_i \right) + \\
&\quad + 2 s_{ac} s_{b\ell_c} \sum_{i=NS,S} \left(\hat{c}^{(acb)def}_i \hat{c}^{(cab)def}_i + \hat{c}^{(acb)def}_i \hat{c}^{(cba)def}_i + \hat{c}^{(acb)def}_i \hat{c}^{(bca)def}_i \right),
\end{aligned} \tag{6.18}$$

for all choices of helicities. Expressing the box coefficients in terms of $N=4$ super-Yang-Mills box-coefficients is useful because it allows one to exploit the recent progress in computing such coefficients however it may not always give the most compact realisation of the supergravity box coefficients. If our ansatz that only box integrals contribute is correct, these coefficients give the complete NMHV $\mathcal{N}=8$ one-loop six-point amplitudes.

6.2 Sample seven- and eight-point box coefficients

We can use the generalised cuts together with the KLT relationship to generate any supergravity or gravity amplitude from the equivalent Yang-Mills amplitudes. For example consider one of the three-mass boxes of the seven-point amplitude,

$$\begin{aligned}
&\hat{c}^{(4^+5^+)(2^-3^-)(6^+7^+)1^-}_{\mathcal{N}=8} = \\
&= 2 s_{23} s_{45} s_{67} \hat{c}^{(4^+5^+)(2^-3^-)(6^+7^+)1^-}_{\mathcal{N}=4}\, \hat{c}^{(5^+4^+)(3^-2^-)(7^+6^+)1^-}_{\mathcal{N}=4} \\
&= \frac{i}{2} \frac{s_{45} s_{23} s_{67} \Big(\langle 2\,3\rangle \langle 1^-|K_{67} K_{45}|1^+\rangle \Big)^8 \langle 5\,2\rangle \langle 3\,6\rangle \langle 4\,3\rangle \langle 2\,7\rangle \,(K^2_{671} K^2_{145} - s_{45} s_{67})^2}{N(1452367) N(1543276) s_{23}^2 \prod_{j=2,3,6,7} \langle 1^-|K_{45} K_{23}|j^+\rangle \prod_{j=2,3,4,5} \langle 1^-|K_{67} K_{23}|j^+\rangle}\,,
\end{aligned} \tag{6.19}$$

with $N(abcdemn) = \langle a\,b\rangle \langle b\,c\rangle \ldots \langle n\,a\rangle$. For this configuration the $\mathcal{N}=4$ box-coefficients are entirely from singlet contributions. We used the results of refs. [27, 28] to substitute for the explicit values of the $\mathcal{N}=4$ gauge theory coefficients. We shall use this result later when we explore the twistor structure of the NMHV amplitudes.

A sample coefficient in the eight-point amplitude is

$$\hat{c}^{(1^-2^-)(5^+6^+)(3^-4^-)(7^+8^+)}_{\mathcal{N}=8} = 2 s_{12} s_{34} s_{45} s_{56}\, \hat{c}^{(1^-2^-)(5^+6^+)(3^-4^-)(7^+8^+)}_{\mathcal{N}=4}\, \hat{c}^{(2^-1^-)(6^+5^+)(4^-3^-)(8^+7^+)}_{\mathcal{N}=4}\,, \tag{6.20}$$

which gives us an example of an N^{2}MHV box coefficient. The $\mathcal{N}=4$ coefficients in this formula may be obtained using the results of ref. [30].

Depending on how we decide to write out the gravity coefficients in terms of Yang-Mills trees, we can have different forms of $\mathcal{N}=8$ box coefficients. As a specific example of this is, an alternative form of the coefficient is, e.g.,

$$\hat{c}^{(1^-2^-)(5^+6^+)(3^-4^-)(7^+8^+)}_{\mathcal{N}=8} = 2 s_{12} s_{34} s_{45} s_{56} \hat{c}^{(1^-2^-)(6^+5^+)(3^-4^-)(8^+7^+)}_{\mathcal{N}=4}\, \hat{c}^{(2^-1^-)(5^+6^+)(4^-3^-)(7^+8^+)}_{\mathcal{N}=4}\,. \tag{6.21}$$

It is interesting to note that the equivalence of the different forms implies a quadratic identity amongst the $\mathcal{N}=4$ box-coefficients.

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6.3 A sample gravity n-point coefficient

In this section we present the computation of an n-point sample term to illustrate the general process. We consider the specific case of the one-mass box $I^{(4^+\cdots n^+)1^-2^-3^-}$. The quadruple cut in this case has the single, singlet, solution

$$\begin{aligned}
\hat{c}^{(4^+,\ldots,n^+)1^-2^-3^-}_{\mathcal{N}=8} &= \frac{1}{2} M^{\rm tree}(4^+,\ldots,n^+,\ell_n^-,-\ell_3^-) \times M^{\rm tree}(1^-,-\ell_n^+,\ell_1^-) \times \\
&\times \times M^{\rm tree}(2^-,-\ell_1^+,\ell_2^+) \times M^{\rm tree}(3^-,-\ell_2^-,\ell_3^+)\,. \qquad (6.22)
\end{aligned}$$

For definiteness, consider the case where $n = 2m+3$ whence we can use the following expression for the KLT relationships [13],

$$\begin{aligned}
M^{\rm tree}(-\ell_3^-,4^+,\ldots,n^+,\ell_n^-) &= -i\Bigg[A^{\rm tree}(-\ell_3^-,4^+,\ldots,n^+,\ell_n^-) \times \\
&\times \sum_{\alpha\in\mathcal{S}_m,\beta\in\mathcal{S}_{m-1}} f(\alpha)\bar{f}(\beta) \times \\
&\times A^{\rm tree}(\alpha_1,\ldots,\alpha_m,-\ell_3,n,\beta_1,\ldots\beta_{m-1},\ell_n)\Bigg] + \\
&+\mathcal{P}(4,\ldots,n-1)\,. \qquad (6.23)
\end{aligned}$$

The α is a permutation of the m legs $(4,\cdots m+3)$ and β is a permutation of the $m-1$ legs $(m+4,\ldots n-1)$. The functions $f(\alpha)$ and $\bar{f}(\beta)$ are polynomial in momenta with

$$\begin{aligned}
f(\alpha) &= s(\ell_3,\alpha_m)\prod_{r=1}^{m-1}\left(s(\ell_3,\alpha_r)+\sum_{k=r+1}^{m} g(\alpha_r,\alpha_k)\right), \\
\bar{f}(\beta) &= s(\beta_1,n)\prod_{r=2}^{m-1}\left(s(\beta_r,n)+\sum_{k=1}^{r-1} g(\beta_k,\beta_r)\right), \qquad (6.24)
\end{aligned}$$

with

$$g(i,j) = \begin{pmatrix} s(i,j)\equiv s_{ij},\ i>j, \\ 0\,,\quad \text{otherwise.}\end{pmatrix}. \qquad (6.25)$$

Solving the on-shell conditions means we substitute

$$\begin{aligned}
s(a,\ell_3) &= \frac{s_{12}\langle 4^+|a|2^+\rangle}{[4\,1]\,\langle 1\,2\rangle} - s_{a3}\,, \\
s(a,\ell_n) &= \frac{s_{23}\langle 1^+|a|2^+\rangle}{[1\,3]\,\langle 3\,2\rangle} + s_{a1}\,. \qquad (6.26)
\end{aligned}$$

Before combining into $\mathcal{N}=8$ coefficients we must reorganise

$$\begin{aligned}
A^{\rm tree}(\alpha_1,\ldots,\alpha_m,-\ell_3,n,\beta_1,\ldots\beta_{m-1},\ell_n) &= A^{\rm tree}(\ell_n,\alpha_1,\ldots,\alpha_m,-\ell_3,n,\beta_1,\ldots\beta_{m-1}) \qquad (6.27)\\
&= -\sum_{\sigma\in OP\{\alpha\}\{\beta^T\cup n\}} A^{\rm tree}(\ell_n,\sigma(\{\alpha\}\{\beta^T\cup n\}),-\ell_3)\,,
\end{aligned}$$

so that

$$M^{\rm tree}(-\ell_3^-,4^+,\ldots,n^+,\ell_n^-) = -i\Bigg[A^{\rm tree}(-\ell_3^-,4^+,\ldots,n^+,\ell_n^-) \times$$

$$\times \sum_{\alpha\in\mathcal{S}_m,\beta\in\mathcal{S}_{m-1}} f(\alpha)\bar{f}(\beta)\times$$
$$\times \sum_{\sigma\in OP\{\alpha\}\{\beta^T\cup n\}} A^{\rm tree}(\ell_n,\sigma(\{\alpha\}\{\beta^T\cup n\}),-\ell_3)\Big]+$$
$$+\mathcal{P}(4,\ldots,n-1)\,. \tag{6.28}$$

Using this we obtain

$$\begin{aligned}\hat{c}^{(4^+,\ldots,n^+)1^-2^-3^-}_{\mathcal{N}=8} &= 2\sum_{\alpha\in\mathcal{S}_m,\beta\in\mathcal{S}_{m-1}} f(\alpha)\bar{f}(\beta)\times\\ &\times\sum_{\sigma\in OP\{\alpha\}\{\beta^T\cup n\}} \hat{c}^{(4^+,\ldots,n^+)1^-2^-3^-}_{\mathcal{N}=4}\,\hat{c}^{(\sigma(\{\alpha\}\{\beta^T\cup n\}))3^-2^-1^-}_{\mathcal{N}=4}+\\ &+\mathcal{P}(4,\ldots,n-1)\,.\end{aligned} \tag{6.29}$$

The explicit forms of these $\mathcal{N}=4$ coefficients may be found in refs. [36, 27, 28]. The explicit form of the gravity coefficient is then obtained by substituting these into the expression.

7. Twistor-space structure of gravity tree amplitudes

The twistor-space properties of supergravity amplitudes are more complicated to analyse than the corresponding Yang-Mills amplitudes since their support is on derivatives of δ-functions rather than simple δ-functions. While the MHV Yang-Mills amplitudes can be seen to have a simple δ-function support

$$\begin{aligned}&A^{\rm tree\,MHV}_n(1^+,2^+,\ldots,p^-,\ldots,q^-,\ldots,n^+)(\lambda_i,\mu_i)\sim\\ &\sim\int d^4x\prod_{i=1}^n\delta^2(\mu_{i\dot a}+x_{a\dot a}\lambda_i^a)A^{\rm tree\,MHV}_n(1^+,2^+,\ldots,p^-,\ldots,q^-,\ldots,n^+)(\lambda_i)\,,\end{aligned} \tag{7.1}$$

where

$$A^{\rm tree\,MHV}_n(1^+,2^+,\ldots,p^-,\ldots,q^-,\ldots,n^+) = i\frac{\langle p\,q\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\cdots\langle n-1,\,n\rangle\,\langle n\,1\rangle}\,, \tag{7.2}$$

is the Parke-Taylor [59] for MHV amplitudes. Expressions for the gravity MHV amplitudes have been presented using the KLT relationship together with factorisation by Berends, Giele and Kuijf [2]. Gravity MHV amplitudes will have a derivative of a δ-function support which mathematically can be expressed as

$$M^{\rm tree\,MHV}_n(1^+,2^+,\ldots,p^-,\ldots,q^-,\ldots,n^+)(\lambda_i,\mu_i)\sim\int d^4xP\Big(-i\frac{\partial}{\partial\mu_{i\dot a}}\Big)\prod_{i=1}^n\delta^2(\mu_{i\dot a}+x_{a\dot a}\lambda_i^a)\,, \tag{7.3}$$

following [17] where P is a polynomial function. This has the geometric meaning that points are sitting infinitesimally off lines in twistor space.

Consequently, we must test the twistor-space behaviour of gravity amplitudes by acting multiple times with the F_{ijk} and K_{ijkl} operators which we define as

$$[F_{ijk},\eta] = \langle i\,j\rangle\left[\frac{\partial}{\partial\tilde\lambda_k},\eta\right]+\langle j\,k\rangle\left[\frac{\partial}{\partial\tilde\lambda_i},\eta\right]+\langle k\,i\rangle\left[\frac{\partial}{\partial\tilde\lambda_j},\eta\right]\,, \tag{7.4}$$

and

$$K_{ijkl} = \frac{1}{4}\left[\langle ij\rangle\epsilon^{\dot a\dot b}\frac{\partial}{\partial\tilde\lambda_k^{\dot a}}\frac{\partial}{\partial\tilde\lambda_l^{\dot b}} - \langle ik\rangle\epsilon^{\dot a\dot b}\frac{\partial}{\partial\tilde\lambda_j^{\dot a}}\frac{\partial}{\partial\tilde\lambda_l^{\dot b}} + \langle il\rangle\epsilon^{\dot a\dot b}\frac{\partial}{\partial\tilde\lambda_j^{\dot a}}\frac{\partial}{\partial\tilde\lambda_k^{\dot b}} + \right.$$
$$\left. + \langle jk\rangle\epsilon^{\dot a\dot b}\frac{\partial}{\partial\tilde\lambda_i^{\dot a}}\frac{\partial}{\partial\tilde\lambda_l^{\dot b}} + \langle jl\rangle\epsilon^{\dot a\dot b}\frac{\partial}{\partial\tilde\lambda_k^{\dot a}}\frac{\partial}{\partial\tilde\lambda_i^{\dot b}} - \langle kl\rangle\epsilon^{\dot a\dot b}\frac{\partial}{\partial\tilde\lambda_j^{\dot a}}\frac{\partial}{\partial\tilde\lambda_i^{\dot b}}\right]. \tag{7.5}$$

For Yang-Mills tree amplitudes the MHV amplitudes satisfy collinearity conditions

$$F_{ijk}A_n^{\text{tree MHV}}(1\ldots n) = 0\,, \tag{7.6}$$

and the NMHV satisfy coplanarity relations

$$K_{ijkl}A_n^{\text{tree NMHV}}(1\ldots n) = 0\,. \tag{7.7}$$

These conditions are manifest in the CSW construction [24] for gauge theory amplitudes. Although, as yet, a similar construction does not exist for gravity, one can determine the twistor-space properties by direct computation. Such an investigation may help shed light on the origin of the difficulties encountered in finding MHV vertices for gravity or finding a string theory dual. Although, Yang-Mills MHV tree amplitudes are holomorphic (independent of $\tilde\lambda_{\dot a}$), the KLT relationships imply that the gravity MHV tree amplitudes are polynomial in $\tilde\lambda_{\dot a}$. From the degree of the polynomial we are guaranteed that

$$F^P M_n^{\text{tree MHV}}(1\ldots n) = 0\,, \qquad \text{for } P > 2(n-3)\,. \tag{7.8}$$

We now show that actually fewer powers of F are required to annihilate the tree amplitudes. We also examine the coplanarity properties of the NMHV amplitudes using the coplanar operator K_{ijkl}.

Consider first the five-point amplitude. There are two inequivalent amplitudes: the MHV and the googly-MHV. These satisfy [47]

$$K^2 M_5^{\text{tree MHV}} = 0\,, \qquad KK' M_5^{\text{tree MHV}} \neq 0\,, \qquad KK'K'' M_5^{\text{tree MHV}} = 0\,, \tag{7.9}$$

and

$$K^2 M_5^{\text{tree googly}} = KK' M_5^{\text{tree googly}} = 0\,, \tag{7.10}$$

where K, K' and K'' represent distinct K_{ijkl}.

By examining the tree amplitudes explicitly we have

$$\begin{aligned} F_{ijk}^4 M_6^{\text{tree MHV}} &= 0\,,\\ F_{ijk}^5 M_7^{\text{tree MHV}} &= 0\,,\\ F_{ijk}^6 M_8^{\text{tree MHV}} &= 0\,, \end{aligned} \tag{7.11}$$

and

$$\begin{aligned} K_{ijkl}^3 M_6^{\text{tree}\,(---+++)} &= 0\,,\\ K_{ijkl}^4 M_7^{\text{tree}\,(---++++)} &= 0\,. \end{aligned} \tag{7.12}$$

These were checked by using computer algebra and by numerically evaluating the expressions at arbitrary kinematic points. The rapid proliferation of terms in the gravity amplitudes as the number of external legs increases makes further checks problematic. In any case, this leads us to postulate the general behaviour,

$$\begin{aligned} F_{ijk}^{n-2} M_n^{\text{tree MHV}} &= 0\,, \\ K_{ijkl}^{n-3} M_n^{\text{tree NMHV}} &= 0\,. \end{aligned} \tag{7.13}$$

8. Twistor-space structure of gravity box coefficients

In this section we show how the box coefficients of supergravity one-loop amplitudes inherit a twistor-space structure directly from box coefficients of super-Yang-Mills theory. In particular, we show that the box-coefficients for MHV gravity amplitudes have collinear support whilst the box-coefficients of NMHV gravity amplitudes have coplanar support, similar to the situation for gauge theory.

Unitarity links the tree amplitudes to the imaginary parts of loop amplitudes. For example considering the cut in a one-loop amplitude we have

$$\begin{aligned} C_{i,\ldots,j} &\equiv \frac{i}{2}\int d\text{LIPS}\Big[M^{\text{tree}}(\ell_1, i, i+1, \ldots, j, \ell_2) \times \\ &\qquad\qquad \times M^{\text{tree}}(-\ell_2, j+1, j+2, \ldots, i-1, -\ell_1)\Big] \\ &= \text{Im}_{K_{i,\ldots,j}>0}\, M^{\text{1-loop}} = \sum_a c_a\, \text{Im}_{K_{i,\ldots,j}>0}(F_a)\,, \end{aligned} \tag{8.1}$$

where the one-loop amplitude is expressed as a sum of integral functions F_a multiplied by rational functions c_a. One can use this expression, and more generalised unitarity expressions to deduce information on the behaviour of the c_a. Specifically, consider the action of a differential twistor space operator $\mathcal{O}$ which satisfies

$$\mathcal{O} M^{\text{tree}}(\ell_1, i, i+1, \ldots, j, \ell_2) = 0\,, \tag{8.2}$$

and where $\mathcal{O}$ only depends on legs $i, \ldots, j$. Naively, the action of $\mathcal{O}$ on the cut gives zero however due to the "holomorphic anomaly" [34, 36] the action of $\mathcal{O}$ produces a δ-function within the integral of the cut leading to [35, 36]

$$\mathcal{O} C_{i,\ldots,j} = \text{rational}\,, \tag{8.3}$$

after the integral has been performed. For the case where F_a is a box integral function the imaginary parts are logarithms of the momentum invariants. In general,

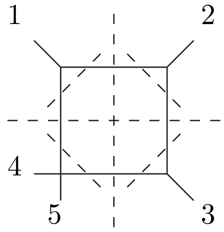
$$\mathcal{O} C_{i,\ldots,j} = \mathcal{O}\Big(\sum_a c_a\, \text{Im}_{K_{i,\ldots,j}>0}(F_a)\Big)\,, \tag{8.4}$$

can only be satisfied provided [36]

$$\mathcal{O} c_a = 0\,. \tag{8.5}$$

We shall apply operators of the form $\mathcal{O} = F^n$ and $\mathcal{O} = K^n$ to the box integral coefficients, using generalised unitarity as a guide to the expected properties.

The first example we will look at is a box-coefficient of the five-point MHV one-loop amplitude. (The action of F_{ijk} does not depend on how the helicities are assigned since the arrangement of helicities will only change a holomorphic factor in the box coefficient.) Consider the various cuts of the box,

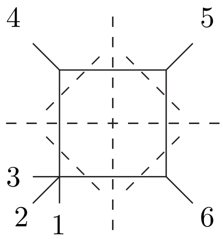


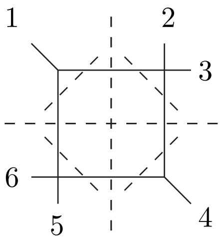
Consider first the cut C_{123}, where the gravity MHV five-point tree amplitude $M^{\rm tree}(\ell_5, 1, 2, 3, \ell_3)$ is isolated on one side of the cut. This tree is annihilated by F^3_{123}. Hence we conclude that the same property should hold for the coefficient: $F^3_{123} c^{(45)123}_{\mathcal{N}=8} = 0$. Similarly by examining the cuts C_{451} and C_{345}, which isolate legs $4, 5, 1$ and $3, 4, 5$ respectively, we deduce that $F^3_{145} c^{(45)123}_{\mathcal{N}=8} = F^3_{345} c^{(45)123}_{\mathcal{N}=8} = 0$. For the remaining choices of F_{ijk} we must consider more generalised cuts. For example in the case of F_{124} we can consider the cut C_{4512}. By analytically continuing to signature $(--++)$ such cuts are possible and will be non-vanishing [30] and allow us to deduce information on the coefficients. In this case the gravity tree amplitude $M^{\rm tree}(\ell_3, 4, 5, 1, 2, \ell_2)$ is a six-point MHV tree, annihilated by, e.g., F^4_{124} and we deduce $F^4_{124} c^{(45)123}_{\mathcal{N}=8} = 0$. Summarising we have

$$F^3_{123} c^{(45)123}_{\mathcal{N}=8} = F^3_{145} c^{(45)123}_{\mathcal{N}=8} = F^3_{345} c^{(45)123}_{\mathcal{N}=8} = 0 \,, \qquad F^4_{ijk} c^{(45)123}_{\mathcal{N}=8} = 0 \quad \forall\, i, j, k \,. \tag{8.6}$$

A similar conclusion holds for all other box coefficients in the five point amplitude, simply by permuting the legs.

Considering the six-point MHV-amplitude there are two types of boxes to consider: the one mass and two-mass-easy boxes,





and examining the various cuts as before we find

$$\begin{aligned} F^3_{ijk} c^{\rm 1m}_{\mathcal{N}=8} &= 0 \,, && \{i, j, k\} = \{1, 2, 3\},\ \{4, 5, 6\} \,, \\ F^4_{ijk} c^{\rm 1m}_{\mathcal{N}=8} &= 0 \,, && \text{if } \{i, j, k\} \in \{3, 4, 5, 6\} \quad \text{or} \quad \{i, j, k\} \in \{1, 4, 5, 6\} \,, \\ F^5_{ijk} c^{\rm 1m}_{\mathcal{N}=8} &= 0 \,, && \forall \{i, j, k\} \,, \end{aligned} \tag{8.7}$$

as well as

$$\begin{aligned}
F^3_{ijk} c^{\rm 2me}_{\mathcal{N}=8} &= 0\,, \qquad \{i,j,k\} = \{1,2,3\},\ \{4,5,6,\},\ \{1,5,6\},\ \{2,3,4\}\,,\\
F^4_{ijk} c^{\rm 2me}_{\mathcal{N}=8} &= 0\,, \qquad \text{if } \{i,j,k\} \in \{1,2,3,4\} \quad \text{or} \quad \{i,j,k\} \in \{1,4,5,6\}\,,\\
F^5_{ijk} c^{\rm 2me}_{\mathcal{N}=8} &= 0\,, \qquad \forall \{i,j,k\}\,,
\end{aligned} \tag{8.8}$$

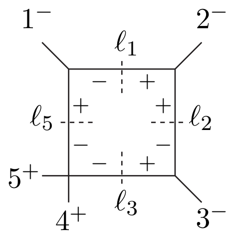
where $c^{\rm 1m}_{\mathcal{N}=8}$ and $c^{\rm 2me}_{\mathcal{N}=8}$ are shorthands for the one-mass and two-mass-easy box coefficients $c^{(123)456}_{\mathcal{N}=8}$ and $c^{1(23)4(56)}_{\mathcal{N}=8}$ As we can see, the box-coefficients have "derivative of δ-function" collinear support although the degree of annihilation by F^n depends on the choice of indices. As before the properties of other coefficients appearing in the amplitude can be obtained simply by permuting labels.

Continuing in this way, by inspecting the general n-point case, we can predict

$$F^{n-1}_{ijk} c^{n\text{-point}}_{\mathcal{N}=8} = 0\,, \qquad \forall i,j,k\,, \tag{8.9}$$

indicating collinearity in twistor space. However, in some cases the value of $n-1$ is not optimal in the sense that smaller multiples of F_{ijk} will suffice for some values of $\{i,j,k\}$.

We now consider the action of K^N on the box-coefficients of the NHMV amplitude $M(1^-,2^-,3^-,4^+,5^+)$ (which is the parity dual of a MHV amplitude). To act with K^N_{ijkl}, we must use a cut which isolates at least four external legs. Specifically, to analyse the behaviour of K^N_{2345} we can examine the C_{2345} cut, where one of the tree amplitudes is $M^{\rm tree}(\ell_1^h, 2^-, 3^-, 4^+, 5^+, \ell_5^{h'})$. If we only consider this cut then the values of h and h' may both be negative and we have a tree amplitude with four negative helicities: such amplitudes do not have coplanar support. However if we consider the quadruple generalised cut then the only possible non-vanishing solution is

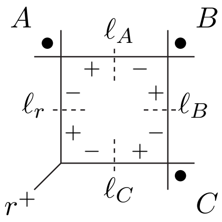


indicating that the problem helicity does not contribute to this box coefficient. The possible tree amplitude $M^{\rm tree}(\ell_1^+, 2^-, 3^-, 4^+, 5^+, \ell_5^-)$ is indeed annihilated by K^3_{2345} which thus requires $K^3_{2345} c_{\mathcal{N}=8} = 0$. Explicit computation confirms that $K^3_{ijkl} c_{\mathcal{N}=8} = 0$ for this box-coefficient.

Continuing in this way we can deduce that at least,

$$K^{n-2} c^{n\text{-point}}_{\mathcal{N}=8} = 0\,, \tag{8.10}$$

although in some cases we need to apply less powers of K. We can consider a generic box with three massless legs, e.g., with exactly one negative helicity on each legs and at least three legs attached to each massive vertex,



where A, B and C are sets of indices of external lines including a single negative helicity. Immediately we can deduce that

$$F^N_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{i,j,k\} \in A \cup \{r\}\,, \qquad F^M_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{i,j,k\} \in B\,,$$
$$F^P_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{i,j,k\} \in C \cup \{r\}\,, \quad (8.11)$$

where N, M and P are integers depending on the number of legs attached to the corner, and

$$K^S_{ijkl} c_{\mathcal{N}=8} = 0 \quad \{i,j,k,l\} \in A \cup B\,, \qquad K^T_{ijkl} c_{\mathcal{N}=8} = 0 \quad \{i,j,k,l\} \in B \cup C\,. \qquad (8.12)$$

The important point here is that the supergravity three-mass box-coefficient in twistor space has a topology inherited from that of super-Yang-Mills: the points lie on three intersecting lines with two of these lines intersecting at point r, as shown in figure 4. This picture is identical to that for super-Yang-Mills [28]–essentially because the argument leading to it is identical albeit with the important difference that we must act with multiple copies of F and K. This example illustrates how the twistor picture for $\mathcal{N} = 8$ box-coefficients will be inherited from that of $\mathcal{N} = 4$ super-Yang-Mills.

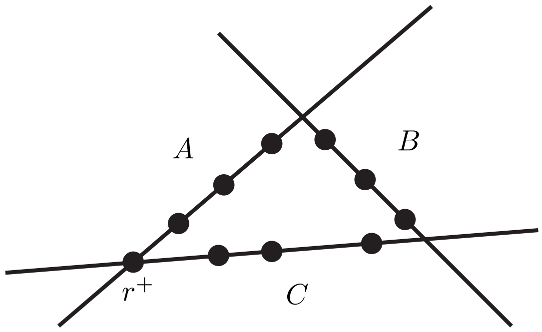


Figure 4: The three-mass box coefficients of the NMHV one-loop amplitude have derivative of δ-function support in twistor space on three intersecting lines lying in a plane. This diagram is identical to the one found in ref. [28] for the NMHV amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory.

As a check of the predicted coplanarity for the $\mathcal{N} = 8$ supergravity box coefficients, we have explicitly computed the action of the K^n operators on the six-point box coefficients. Using computer algebra, and numerically evaluating the results at a generic kinematic point we have verified that all gravity six-point coefficients have a derivative of a δ-function support on planes in twistor space, confirming the predicted patterns.

The twistor support of the four-mass box-coefficients of the N^2MHV gravity amplitudes is also very similar to the Yang-Mills N^2MHV case [28, 30]. If we consider the generic four-

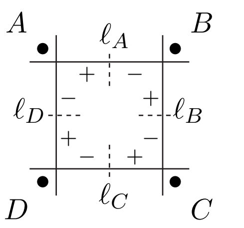


Figure 5: A four-mass box with clusters of legs indicated by A, B, C, D. The dots represent arbitrary numbers of external legs.

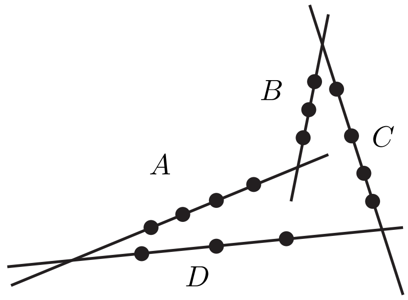


Figure 6: The four-mass box-coefficients of N^{2}MHV amplitudes have derivative of δ-function support in twistor space on four intersecting lines.

mass box with clusters of legs A, B, C and D as indicated in figure 5, for the N^{2}MHV case the clusters form four MHV trees. From this observation we obtain,

$$\begin{aligned}
&F^{r}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in A \qquad F^{r'}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in B\,, \\
&F^{r''}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in C \qquad F^{r'''}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in D\,,
\end{aligned} \tag{8.13}$$

and

$$\begin{aligned}
&K^{s}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in A \cup B\,, \qquad K^{s'}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in B \cup C\,, \\
&K^{s''}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in C \cup D\,, \qquad K^{s'''}_{ijk} c_{\mathcal{N}=8} = 0\,, \quad \{ijk\} \in D \cup A\,,
\end{aligned} \tag{8.14}$$

which gives us a picture of four pair-wise intersecting lines or of points lying on a pair of intersecting planes shown in figure 6. Again this matches the picture for $\mathcal{N} = 4$ super-Yang-Mills theory [28], except that the lines have derivative of δ-function support.

In summary, the box-coefficients of one-loop amplitudes in $\mathcal{N} = 8$ supergravity have a twistor-space structure inherited from $\mathcal{N} = 4$ super-Yang-Mills amplitudes.

9. Conclusions

Gauge and gravity theories are two of the cornerstones of modern theoretical physics. Explicit calculations within these theories have been very fruitful for uncovering and testing theoretical properties. In the gauge theory case, such calculations are also crucial for comparisons of theory against experiments.

In this paper we have investigated the twistor-space properties of both tree and loop amplitudes in $\mathcal{N} = 8$ supergravity which is especially interesting because of its close connection to $D = 11$ supergravity, which again is closely linked to M-theory. It is also believed to be the gravity theory with the best ultraviolet properties: the first potential divergence occurs at no less than five loops [15, 16].

In general, gravity tree amplitudes are simple to obtain via the low energy limit of the KLT relationship [1 − 3] between open and closed strings. In the case of loop amplitudes the technically simplest theory to deal with is that of $\mathcal{N} = 8$ supergravity. Here we have computed the coefficients of the box functions in $\mathcal{N} = 8$ supergravity in order to determine their twistor-space properties.

Loop calculations in quantum gravity theories are notoriously difficult: direct calculation using Feynman diagram techniques being significantly more difficult than the equivalent gauge theory ones. In this paper we followed the logic of refs. [15, 12, 13], making use of the unitarity method together with the KLT relations to obtain supergravity loop amplitudes from gauge theory tree amplitudes. The observation of ref. [30] that quadruple cuts freeze the loop integrals helped simplify our discussion.

We have produced sample formulæ for supergravity box integral coefficients from known gauge theory amplitudes. Following similar logic, it should be possible to obtain the full n-point gravity one-loop amplitudes by recycling the equivalent Yang-Mills gauge theory amplitudes. Specifically we have calculated box integral coefficients in $\mathcal{N} = 8$ supergravity using the corresponding ones of $\mathcal{N} = 4$ super-Yang-Mills theory. In many ways these box coefficients are simpler objects than tree amplitudes. As has happened for gauge theory [27, 28, 60, 29, 61], it seems likely, although perverse, that loop amplitudes will prove a route to simplifying gravity tree amplitudes.

Factorisation properties suggest that only box integral functions appear in one-loop $\mathcal{N} = 8$ supergravity amplitudes. If this is true, one can obtain complete one-loop $\mathcal{N} = 8$ supergravity amplitudes simply by evaluating the box integral coefficients. This, however, would require a non-trivial cancellation, since the gravity amplitudes are *a priori* not cut-constructible from the four-dimensional tree amplitudes. Rather surprisingly, it also seems to imply that at one-loop $\mathcal{N} = 8$ supergravity has a power counting identical to that of $\mathcal{N} = 4$ super-Yang-Mills theory. It would be very interesting to prove whether box function are the only integral functions that appear in $\mathcal{N} = 8$ one-loop amplitudes. Assuming a proof is found, it would also be very interesting to investigate possible implications of these types of cancellations on the higher-loop ultraviolet divergences of $\mathcal{N} = 8$ supergravity.

We also showed how twistor space properties of gauge theory amplitudes are inherited by gravity loop amplitudes, via the Kawai-Lewellen and Tye tree relations and the unitarity method. The relatively simple twistor-space structure of the gravity amplitudes described in this paper hints that there may be a twistor-space string theory interpretation. We hope that further investigations will provide new insight into quantum gravity.

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